

Lack of thermalisation in a Fermi liquid

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(Dated: March 25, 2014)

We study an interaction quench in a three-dimensional Fermi gas. We first show that the perturbative expansion of the long-wavelength structure factor $S(\mathbf{q})$ is not compatible with the hypothesis that steady-state averages correspond to thermal ones. In particular, $S(\mathbf{q})$ remains non-analytic at $\mathbf{q} \rightarrow \mathbf{0}$, signaling a power-law decay of correlation functions in real space instead of the finite-temperature exponential one. We next consider the case of a dilute gas, where one can obtain non-perturbative results in the interaction strength but at lowest order in the density. We find that in the steady-state the momentum distribution jump at the Fermi surface remains finite, though smaller than in equilibrium, up to second order in $k_F f_0$, where f_0 is the scattering length of two particles in the vacuum. Both results question the emergence of a finite length scale in the unitary quench-dynamics as expected by thermalisation.

PACS numbers: 71.10.-w, 05.30.Fk, 05.70.Ln

The past few years have witnessed a growing interest in quantum non-equilibrium phenomena,[1] mostly motivated by cold-atom[2] and ultrafast pump-probe spectroscopy experiments.[3] Among the issues under discussion, one of the most important is whether a macroscopic but isolated quantum system can serve as its own dissipative bath. Imagine that such a system is initially supplied with an extensive excess energy and then let evolve until it relaxes to a steady state. Should quantum ergodicity hold,[4, 5] the steady-state values of local observables would coincide with thermal averages at an effective temperature T_* such that the internal energy coincides with the initial one, which is conserved during the unitary evolution. The problem under discussion is when and how the above *thermalisation hypothesis* holds.[1, 6, 7]

Experiments on physical realizations of almost integrable models [8, 9] do not actually find evidences of thermalisation *within accessible time scales*. The common wisdom is that such non-thermal state, sometimes referred to as *pre-thermalisation* and observed in a wealth of different model calculations,[10–13] even far from integrability,[14–17] will eventually give in to a thermal state at long times. However, the final flow towards thermal equilibrium remains so far rather elusive. Numerically exact calculations exist but are not conclusive. For instance, simulations of an interaction quench in the infinite-dimensional half-filled Fermi-Hubbard model[15] do show that the energy distribution jump at the Fermi energy, after a plateau compatible with pre-thermalization,[14] starts a steady decrease at longer times. Unfortunately, the affordable simulation time is too short to ascertain whether the jump does indeed relax to zero as expected by thermalisation, especially at interaction values safely low to exclude any influence by the Mott transition. Thermalization can be found by employing the pre-thermal state as initial condition of a Boltzman-type equation for quasiparticles,[18]

which is unsurprising given the premise of a collision integral that assumes applicability of Wick theorem for quasiparticles and vanishes at thermal equilibrium.

Thermalisation in Fermi liquids is likely to occur by the continuum of gapless excitations that can efficiently dissipate and redistribute the excess energy. It is however evident that thermalisation after an interaction quench in a Fermi liquid is in apparent contradiction with the original Landau construction.[19] Indeed, the idea of a Fermi liquid is inherently based on the expectation that, if interaction is slowly turned on, the non-interacting low-energy eigenstates *adiabatically* evolve into the fully-interacting ones, so that low-lying eigenstates of interacting fermions end to be in one-to-one correspondence to those of free fermions. On the other hand, if the interaction is turned on in a finite time τ , however long, the final-state energy differs from the ground state one by an extensive amount. Should thermalisation indeed occur, such a finite energy density would translate into a finite temperature T_* . Since a Landau-Fermi liquid can be regarded as a quantum-critical state where all correlation functions decay as a power-law in the distance at zero-temperature and exponentially at any $T \neq 0$, this would imply that, no matter how large the interaction switching-time τ is, if thermalisation holds the initial non-interacting Fermi sea will *never* evolve into the interacting ground state. As a matter of fact, the same problem, and, possibly,[20] also the same conclusions we are going to draw, applies to any quantum critical state suddenly perturbed by an irrelevant or marginally-irrelevant operator.

The purpose of this Letter is to contribute to the solution of this puzzle by showing analytically that a weak interaction quench in a Fermi liquid does not lead to thermalisation in some characteristic observables sensitive to the range of correlations, such as the static structure factor and the momentum distribution function. We will do so by introducing a perturbative test for thermalisation,

based on the comparison between thermal expectation values of operators and non-equilibrium stationary state values obtained within perturbation theory. We will then apply this to the static structure factor of a three dimensional Fermi-Hubbard model, showing that indeed the putative exponential decay of the steady-state correlations does not appear at the expected order in perturbation theory, implying lack of thermalisation. Finally, we will show that steady-state correlations remain power-law even when non-perturbative results in the interaction are accessible, specifically in the dilute Fermi gas at leading order in the density.

Let us start our analysis by formulating a general criterion to test the occurrence of thermalisation. Hereafter we shall deal with the specific example of a three-dimensional Fermi-Hubbard model $\mathcal{H} = \mathcal{H}_0 + U\mathcal{U}$, with

$$\mathcal{H}_0 + U\mathcal{U} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{U}{V} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q} \neq \mathbf{0}} c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{p}+\mathbf{q}\downarrow}^\dagger c_{\mathbf{p}\downarrow} c_{\mathbf{k}+\mathbf{q}\uparrow}, \quad (1)$$

where V is the volume and we have removed the $\mathbf{q} = \mathbf{0}$ Hartree-term. Imagine that the system is prepared in the Fermi sea $|0\rangle$, ground state of \mathcal{H}_0 with energy E_0 , and let evolve with the full Hamiltonian (1) with an interaction strength $U \ll 1$. The energy of the system (conserved under the unitary evolution) is $E = \langle 0 | \mathcal{H} | 0 \rangle = E_0$. If the time evolution leads to a thermal state, its internal energy, within second order perturbation theory both in U and in temperature T , reads

$$U(T) = E_0 - U^2 \sum_{n \neq 0} \frac{|\langle n | \mathcal{U} | 0 \rangle|^2}{E_n - E_0} + \frac{V}{2} \gamma_c T^2, \quad (2)$$

where $|n\rangle$ are eigenstates of \mathcal{H}_0 with eigenvalues E_n and γ_c is the non-interacting specific-heat coefficient. We may obtain the effective temperature T_* at which the system would thermalize setting $U(T_*) = E$, thus obtaining

$$T_*(U) \simeq U \left(\frac{2}{\gamma_c V} \sum_{n \neq 0} \frac{|\langle n | \mathcal{U} | 0 \rangle|^2}{E_n - E_0} \right)^{1/2} + O(U^2), \quad (3)$$

which starts therefore linear in U . [18]

Let us now consider the dynamics of an observable \mathcal{O} and in particular its steady state value $O_*(U)$. If thermalization occurs then

$$O_* = O_{eq}(T_*, U), \quad (4)$$

where O_{eq} is the thermal average. If we expand in power series both steady state value and thermal average, i.e. $O_* = \sum_{n \geq 0} O_*^{(n)} U^n$ and $O_{eq.}(U, T_*) = \sum_{n, m \geq 0} O_T^{(n, m)} U^n T_*^m$, we may verify the validity of Eq.(4) order by order in perturbation theory through Eq. (3) evaluated at the required order.

In reality, one can usually access only leading terms in the perturbation series; it is therefore important to

implement the program above focusing on observables that sharply discriminate, already at lowest orders, between thermal and non-thermal behavior. Focusing on a Fermi-gas, one of such observables is the static structure factor $S(\mathbf{q})$, i.e. the Fourier transform of the static density-density correlation function $S(\mathbf{r}) = \langle n_{\mathbf{r}} n_{\mathbf{0}} \rangle - n^2$, where $n_{\mathbf{r}}$ is the density operator and n its average. Indeed one readily realizes (see Supplemental Materials) that its analytic properties for $|\mathbf{q}| \rightarrow 0$ depend whether correlations decay power-law or exponentially at large distances. Specifically, if the Taylor expansion of $S(\mathbf{q})$ near $\mathbf{q} = \mathbf{0}$ contains odd powers, e.g. $|\mathbf{q}|^{2n+1}$, then $S(\mathbf{r})$ decays as a power-law in the distance, specifically $S(\mathbf{r}) \sim 1/r^{2n+4}$. On the contrary, $S(\mathbf{q}) \sim A + O(q^2)$ corresponds to $S(\mathbf{r}) \sim e^{-r/\xi}$, which is indeed what we expect if thermalisation occurs. For instance, for non-interacting electrons $S_0(\mathbf{r}, T) = n \delta_{\mathbf{r}\mathbf{0}} - 2G(\mathbf{r}, T)^2$, where $G(\mathbf{r}, T) = \langle c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{0}\sigma} \rangle_0$, which, for $k_F r \gg 1$ reads

$$G(\mathbf{r}, T) \simeq -\frac{T k_F}{\pi v_F r \sinh(\pi T r / v_F)} \left[\cos(k_F r) - \frac{\pi T}{v_F k_F} \sin(k_F r) \coth(\pi T r / v_F) \right], \quad (5)$$

and decays as $\cos(k_F r)/r^2$ at $T=0$ and exponentially at $T \neq 0$. As a consequence, $S_0(\mathbf{q}, T) = k_F^2 |\mathbf{q}|/6\pi^2$ is non-analytic when $T=0$ at $q \rightarrow 0$ (as well as at $q \sim 2k_F$), but turns analytic as soon as $T \neq 0$, $S_0(\mathbf{q} \rightarrow \mathbf{0}, T) = 2T \rho_0 + O(q^2)$, with ρ_0 the single-particle density of states at the chemical potential. At equilibrium but in the presence of interaction, one can readily show that the leading corrections, both in T and U , are determined solely by temperature. Therefore, if thermalisation indeed holds, then, while initially the structure factor is $S_0(\mathbf{q}, 0)$, hence non-analytic, in the steady state $S_*(\mathbf{q} \rightarrow \mathbf{0}) \simeq S_0(\mathbf{q} \rightarrow \mathbf{0}, T_*(U))$, i.e. its analytic properties should totally change during the unitary evolution. Within perturbation theory, such an analyticity switch must show up as a singularity in the perturbative expansion. Indeed, if we first expand $G(\mathbf{r}, T)$ of Eq. (5) in $T = T_*(U) \sim U$, and only after we Fourier transform, then we would find at small $|\mathbf{q}|$ a second order correction in three dimensions $S_*(\mathbf{q} \rightarrow \mathbf{0}) - S_0(\mathbf{q}) \sim (T_*(U)^2)/|\mathbf{q}| \sim U^2/|\mathbf{q}|$, which is therefore singular as $|\mathbf{q}| \rightarrow 0$.

The occurrence of thermalization after an interaction quench can be readily confirmed by checking whether the second order correction to $S_*(\mathbf{q})$ is singular as $|\mathbf{q}| \rightarrow 0$. This is an elementary though lengthy calculation, which we thoroughly describe in the Supplemental Material. The outcome is however totally unexpected. First of all, we find that the approach to the steady state is perfectly defined within perturbation theory and provided the thermodynamic limit is taken first. For instance, at first order,

$$S(\mathbf{q}, t) \simeq S_0(\mathbf{q}) \quad (6)$$

$$-4U \int_0^\infty d\omega_1 d\omega_2 \rho_{\mathbf{q}}(\omega_1) \rho_{\mathbf{q}}(\omega_2) \frac{1 - \cos(\omega_1 + \omega_2)t}{\omega_1 + \omega_2},$$

where hereafter momentum and energy are in units of k_F and ϵ_F , respectively, $\rho_{\mathbf{q}}(\omega)$ is the density of states of particle-hole excitations at momentum \mathbf{q} . Since for small $|\mathbf{q}|$ and $\omega \leq v_F |\mathbf{q}|$, $\rho_{\mathbf{q}}(\omega) \sim \omega/v_F |\mathbf{q}|$, the structure factor remains $\propto |\mathbf{q}|$ at first order in U , and the time-dependent term in (6) vanishes as a power-law in $1/t$ for large times. In addition, and more remarkably, we find that in the steady-state the second order corrections are not singular as predicted by thermalization. Specifically, $S_*(\mathbf{q}) \sim A + B |\mathbf{q}| + O(q^2)$, where, e.g.

$$A = 16 U^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \iint_0^\infty d\omega_1 d\omega_2 \frac{\rho_{\mathbf{p}}(\omega_1) \rho_{\mathbf{p}}(\omega_2)}{(\omega_1 + \omega_2)^2}, \quad (7)$$

$B = 1/16\pi^2 + O(U)$, so that in real space $S_*(\mathbf{r})$ still has a power-law decaying term that coexists with an exponentially vanishing one arising at order U^2 . We mention that $S_*(\mathbf{q}) \sim A + B |\mathbf{q}|$ may also explain the contradictory one-dimensional results of Ref. [21].

Going beyond leading order is in general unfeasible but in limiting cases where a consistent re-summation of the perturbative series is possible in terms of expansion parameters different from interaction. At equilibrium this occurs for instance in a dense Fermi gas with long-range Coulomb forces, or, alternatively, in the dilute limit with short-range interactions, where Galitskii[22] showed that replacing the full interaction vertex with the ladder diagrams in the particle-particle channel provides results valid at any order in interaction but at leading in the density. This was shown to actually correspond to diagonalizing the Hamiltonian in the subspace that includes, besides the Fermi sea, states with spin-singlet pairs of holes and particles, inside and outside the Fermi sphere, respectively.[23] This approximate scheme is in turn close to Anderson's treatment of quantum fluctuation corrections to the BCS mean-field theory of superconductivity,[24] which we shall exploit to extend out-of-equilibrium Galitskii's theory. Indeed, let us suppose that quantum fluctuations brought by interaction do not spoil completely the non-interacting Fermi sphere, which we shall regard as the *vacuum* of the quantum fluctuations, hypothesis to be verified *a posteriori*. We observe that

$$\begin{aligned} [c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger, c_{-\mathbf{p}+\mathbf{q}\downarrow} c_{\mathbf{p}\uparrow}] &= -\delta_{\mathbf{k}\mathbf{p}} c_{-\mathbf{k}+\mathbf{q}\downarrow} c_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger \\ &+ \delta_{-\mathbf{k}+\mathbf{q}, -\mathbf{p}+\mathbf{q}} c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}+\mathbf{q}'-\mathbf{q}\uparrow}, \end{aligned} \quad (8)$$

has a finite value on the Fermi sea only if $\mathbf{k} = \mathbf{p}$ and $\mathbf{q} = \mathbf{q}'$, in which case is either +1 or -1 depending whether \mathbf{k} and $-\mathbf{k} + \mathbf{q}$ are both inside or outside the Fermi sea, respectively. In the same spirit as e.g. bosonization, we

approximate the r.h.s. of Eq. (8) by its average value on the Fermi sphere. Therefore, we associate to the pair creation operator $c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger$, where \mathbf{k} and $-\mathbf{k} + \mathbf{q}$ are both outside the Fermi sphere, a hard-core boson creation operator $b_{\mathbf{k},\mathbf{q}}^\dagger$. Seemingly, we associate another independent hard-core boson operator $a_{\mathbf{k},\mathbf{q}}^\dagger$ to $c_{-\mathbf{k}+\mathbf{q}\downarrow} c_{\mathbf{k}\uparrow}$, where now both \mathbf{k} and $-\mathbf{k} + \mathbf{q}$ are inside the Fermi sphere. Since $[\mathcal{H}_0, c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger] = (\epsilon_{\mathbf{k}} + \epsilon_{-\mathbf{k}+\mathbf{q}}) c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}\downarrow}^\dagger$, the non-interacting dynamics of the hard-core bosons can be reproduced by mapping $\mathcal{H}_0 \rightarrow \sum_{\mathbf{q}} \sum_{\mathbf{k}} \omega_{\mathbf{k},\mathbf{q}} (a_{\mathbf{k},\mathbf{q}}^\dagger a_{\mathbf{k},\mathbf{q}} + b_{\mathbf{k},\mathbf{q}}^\dagger b_{\mathbf{k},\mathbf{q}})$, where $\omega_{\mathbf{k},\mathbf{q}} = |\epsilon_{\mathbf{k}} + \epsilon_{-\mathbf{k}+\mathbf{q}}| > 0$. In this scheme, the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + U \mathcal{U}$ is thus mapped onto[24]

$$\begin{aligned} \mathcal{H}_* &= \sum_{\mathbf{q}} \mathcal{H}_{\mathbf{q}} = \sum_{\mathbf{q}} \left[\sum_{\mathbf{k}} \omega_{\mathbf{k},\mathbf{q}} (a_{\mathbf{k},\mathbf{q}}^\dagger a_{\mathbf{k},\mathbf{q}} + b_{\mathbf{k},\mathbf{q}}^\dagger b_{\mathbf{k},\mathbf{q}}) \right. \\ &\quad \left. + \frac{U}{V} \sum_{\mathbf{k},\mathbf{p}} (b_{\mathbf{k},\mathbf{q}}^\dagger - a_{\mathbf{k},\mathbf{q}}) (b_{\mathbf{p},\mathbf{q}} - a_{\mathbf{p},\mathbf{q}}^\dagger) \right]. \end{aligned} \quad (9)$$

The vacuum of the hard-core bosons is the Fermi sea, and the role of the interaction, second term on the r.h.s. of Eq. (9), is to create pairs of holes and particles out of the vacuum.

We observe that each $\mathcal{H}_{\mathbf{q}}$ in Eq. (9) resembles the Hamiltonian of hard-core bosons in the presence of a local potential. We thence foresee that $\langle a_{\mathbf{k},\mathbf{q}}^\dagger a_{\mathbf{k},\mathbf{q}} \rangle \sim \langle b_{\mathbf{k},\mathbf{q}}^\dagger b_{\mathbf{k},\mathbf{q}} \rangle \sim 1/V$. Consequently, we expect it is safe to relax the hard-core constraint, and regard $a_{\mathbf{k},\mathbf{q}}$ and $b_{\mathbf{k},\mathbf{q}}$ as conventional bosons. Within this approximation, which we verified *a posteriori*, it is relatively straightforward to diagonalise Eq. (9) (see Supplementary Material).

The Hamiltonian (9) can be exploited to reproduce the dynamical behavior of certain electronic observables following a sudden quench. For instance, the time-evolution of the momentum distribution, through the equation of motion $i\dot{n}_{\mathbf{k}} = [\mathcal{H}, n_{\mathbf{k}}]$, maps onto

$$n_{\mathbf{p}}(t) \simeq \begin{cases} 1 - \sum_{\mathbf{q}} \langle \psi(t) | a_{\mathbf{p},\mathbf{q}}^\dagger a_{\mathbf{p},\mathbf{q}} | \psi(t) \rangle & \text{if } |\mathbf{p}| \leq k_F, \\ \sum_{\mathbf{q}} \langle \psi(t) | b_{\mathbf{p},\mathbf{q}}^\dagger b_{\mathbf{p},\mathbf{q}} | \psi(t) \rangle & \text{if } |\mathbf{p}| > k_F, \end{cases}$$

so that the jump at the Fermi surface

$$Z(t) \simeq 1 - \sum_{\mathbf{q}} \langle \psi(t) | a_{\mathbf{k},\mathbf{q}}^\dagger a_{\mathbf{k},\mathbf{q}} + b_{\mathbf{k},\mathbf{q}}^\dagger b_{\mathbf{k},\mathbf{q}} | \psi(t) \rangle, \quad (10)$$

where $|\mathbf{k}|$ is on the Fermi sphere and $|\psi(t)\rangle$ is the boson vacuum evolved with the Hamiltonian (9). Through the exact diagonalization of the latter, assuming a spherical Fermi surface with energy dispersion $\epsilon_{\mathbf{k}} = k^2$, which is indeed appropriate in the low-density limit, we obtain a steady state value of the jump at the Fermi surface $Z_* = Z_{eq} + \delta Z_*$ (see Supplemental Material), where

$$\delta Z_* = -\frac{(k_F a)^3}{8\pi^2} \int_0^2 q dq \left[\int_0^{q(2-q)} d\omega \left| T(\omega - i0^+, \mathbf{q}) \right|^2 \int_0^2 d\epsilon \frac{\mathcal{N}_{\text{OUT}}(\epsilon, \mathbf{q})}{(\omega + \epsilon)^2} + \int_0^{q(2+q)} d\omega \left| T(-\omega + i0^+, \mathbf{q}) \right|^2 \int_0^2 d\epsilon \frac{\mathcal{N}_{\text{IN}}(\epsilon, \mathbf{q})}{(\omega + \epsilon)^2} \right], \quad (11)$$

a is the lattice spacing and Z_{eq} is the equilibrium value at zero temperature[22]

$$Z_{eq} = 1 - \frac{(k_F a)^3}{8\pi^2} \int_0^2 q dq \left[\int_0^{q(2-q)} d\omega \int_0^2 \frac{d\epsilon}{\pi} \frac{\Im m T(-\epsilon + i0^+, \mathbf{q})}{(\epsilon + \omega)^2} + \int_0^{q(2+q)} d\omega \int_0^2 \frac{d\epsilon}{\pi} \frac{\Im m T(\epsilon - i0^+, \mathbf{q})}{(\epsilon + \omega)^2} \right].$$

The function of complex variable $T(z)$ defined through

$$U T^{-1}(z, \mathbf{q}) = 1 - U \chi(z, \mathbf{q}) = 1 - U \int_0^2 d\epsilon \frac{\mathcal{N}_{\text{IN}}(\epsilon, \mathbf{q})}{z - \epsilon} + U \int_0^2 d\epsilon \frac{\mathcal{N}_{\text{OUT}}(\epsilon, \mathbf{q})}{z + \epsilon}, \quad (12)$$

is just the usual T -matrix, with $\chi(z, \mathbf{q})$ the non-interacting Cooper bubble, $\mathcal{N}_{\text{IN}}(\epsilon, \mathbf{q})$ and $\mathcal{N}_{\text{OUT}}(\epsilon, \mathbf{q})$ the density of states of a pair of holes and particles, respectively, at total momentum \mathbf{q} . In particular, if we expand up to second order in U we find that $1 - Z_* = 2(1 - Z_{eq})$, in agreement with Ref. 14.

Following Galitskii,[22] if $\chi_0(z, \mathbf{q})$ is the Cooper bubble of two electrons in the vacuum, then

$$T(z, \mathbf{q}) \simeq T_0(z, \mathbf{q}) + T_0(z, \mathbf{q})^2 \left(\chi(z, \mathbf{q}) - \chi_0(z, \mathbf{q}) \right), \quad (13)$$

where $T_0(z, \mathbf{q})$ is the scattering T -matrix in the vacuum. Using the first order expansion (13) in Eq. (11), one consistently obtains a value that is exact at any order in the interaction U , but valid up to second order in $T_0(z \rightarrow 0) \sim k_F f_0$, where f_0 is the scattering length of two particles in the vacuum.[22]

We thus find that Z_* in the steady state remains strictly finite, though smaller than at equilibrium, at leading order in the density but infinite in U . The simplest interpretation of this result is that the momentum distribution jump does not thermalize at low density at any order in perturbation theory, which is consistent with the previous second order calculation. Indeed the discontinuity of $n_{\mathbf{k}}$ at the Fermi surface implies a Friedel-like behavior of the steady-state single-particle density matrix $G_*(\mathbf{r}) = \langle c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{0}\sigma} \rangle_{t \rightarrow \infty} \sim \cos(k_F r)/r^2$, at odds with the thermalisation prediction of an exponential decay.

These results question the occurrence of thermalisation in a Fermi gas at least in its broad accepted meaning. Notice, however, that the restriction of the notion of thermalization to local observables would have no contradiction with our findings and no inconsistency as well with the adiabatic assumption at the basis of Landau's Fermi-liquid theory, since the latter mainly concerns long-wavelength properties. Indeed, the long-wavelength limit of the static structure factor or the momentum distribution close to k_F are, rigorously speaking, not "local observables".[25]

We have analyzed the long-distance correlations in the steady-state of a non-interacting Fermi sea unitarily evolved with an interacting Hamiltonian. We have first calculated up to second order in the interaction strength the structure factor, whose analytic behavior close to momentum $|\mathbf{q}| \rightarrow 0$ neatly depends whether real-space correlations decay as a power-law or exponentially. We have found that signals of an exponential decay do not appear up to second order contrary to what would be expected if steady-state averages were equal to equilibrium ones at finite temperature. Next, we have calculated the long-time value Z_* of the Fermi distribution jump at the Fermi surface in the case of a dilute gas, where it is possible to obtain consistent results valid at any order in the interaction strength but at leading order in the density. We have found that Z_* remains finite, though smaller than at equilibrium, which in turn implies that also single-particle correlations remain power-law in the steady state, at least up to leading order in the electron density.

We acknowledge useful discussions with F. Becca and A. Parola. This work has been supported by the European Union, Seventh Framework Programme, under the project GO FAST, Grant Agreement no. 280555.

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- [25] We cannot exclude that the perturbative expansion breaks down at higher orders. This may occur only if higher order terms are singular as $t \rightarrow \infty$, which thence nullifies any perturbative expansion. If the interaction is switched on as $U(t) = U(1 - e^{-\epsilon t})$, with $\epsilon > 0$, one should find higher order terms in perturbation theory that are singular at large times for any $\epsilon \neq 0$ but vanish for $\epsilon \rightarrow 0$, which is well possible but far not trivial.
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Supplemental Material to "Lack of Thermalization in a Fermi Liquid"

In this Supplemental Material we present in detail the calculations that are sketched and discussed in the main text.

REMARKS ON THE LONG-WAVELENGTH STRUCTURE FACTOR $S(\mathbf{q})$

Here we recall the connection between the analytic properties of the long-wavelength structure factor $S(\mathbf{q})$ and the behaviour in real space of its inverse Fourier transform, $S(\mathbf{r})$. In three dimensions and assuming $S(\mathbf{r}) = S(r)$, we find

$$S(\mathbf{q}) = S(q) = \int d\mathbf{r} S(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{4\pi}{q} \int_0^\infty r dr S(r) \sin(qr). \quad (1)$$

Even though $q = |\mathbf{q}|$ is by definition positive, if we extend artificially the domain of $S(q)$ also to $q \leq 0$, we readily note that $S(q) = S(-q)$ in an even function. Therefore the Taylor expansion of $S(q)$ near $q = 0$ must contain even powers, q^{2n} , and/or odd powers but of the absolute value, $|q|^{2n+1}$. In the latter case the function $S(q)$, $q \in [-\infty, +\infty]$, is not analytic at the origin (and presumably also at $q = \pm 2k_F$).

Through Eq. (1), we observe that, if $S(r)$ vanishes faster than any power of $1/r$ for $r \rightarrow \infty$, then only even-order derivatives of $S(q)$ are finite for $q \rightarrow 0$; the function is analytic. On the contrary, if $S(r) \sim 1/r^{2m}$ for large r and $m \geq 2$, then the derivative of order $2m - 3$ will be finite at $q = 0$, i.e. $S(q)$ will have a non-analytic behaviour $S(q) \sim |q|^{2m-3}$ at small q . In particular, if

$$S(r \rightarrow \infty) \sim A e^{-r/\xi} + \frac{B}{r^4},$$

then

$$S(q \rightarrow 0) \sim 8\pi \xi^3 A - \pi^2 B |q| + O(q^2).$$

Therefore, even though the term linear in q is sub-leading with respect to the constant, its inverse Fourier transform corresponds to a power-law decaying contribution, which dominates over the exponentially vanishing one that derives from the leading constant-in- q term $8\pi \xi^3 A$. In other words, what discriminates between a power-law with respect to an exponential decay of $S(r)$ for large r is just the finiteness of odd-order derivatives of $S(q)$ at $q = 0$.

SECOND ORDER PERTURBATION THEORY

We consider a Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + U(t)\mathcal{U}, \quad (2)$$

where the unperturbed \mathcal{H}_0 has eigenstates $|n\rangle$ with eigenvalues E_n , measured with respect to the ground state energy. We shall assume that $\langle n | \mathcal{U} | n \rangle = 0$ for any $|n\rangle$ and, in addition, that the Hamiltonian as well as the eigenvalues are real.

Although we are actually interested only in the sudden quench, we shall here consider a more general turning on of the interaction,

$$U(t) = U (1 - e^{-\epsilon t}), \quad (3)$$

with $\epsilon > 0$, which interpolates between the sudden switch, $\epsilon \rightarrow \infty$, and the adiabatic one, set first $\epsilon t \gg 1$ and then send $\epsilon \rightarrow 0$. The reason is that, as we shall see, there is not a dramatic difference between the adiabatic $\epsilon \rightarrow 0$ limit and the general case of finite ϵ , at least up to second order. Since the former is known not to lead to singularities as $t \rightarrow \infty$, this suggests the same holds for any $\epsilon \neq 0$.

We assume that the system is initially in the ground state $|0\rangle$ of \mathcal{H}_0 , and evolves at positive times with the interacting Hamiltonian (2). We study the time evolution by second order perturbation theory applied directly to the Schrödinger equation in the interaction representation. Namely we write the wavefunction as

$$|\Psi(t)\rangle = e^{-i\mathcal{H}t} |0\rangle = e^{-i\mathcal{H}_0 t} |\Phi(t)\rangle, \quad (4)$$

where $|\Phi(0)\rangle = |0\rangle$, and set $|\Phi(t)\rangle = \sum_{m \geq 0} U^m |\phi_m(t)\rangle$, where m is the order in perturbation theory, being $|\phi_0(t)\rangle = |0\rangle$ and, for any $m > 0$, $|\phi_m(0)\rangle = 0$. It readily follows that

$$i \partial_t |\phi_m(t)\rangle = \frac{U(t)}{U} e^{i\mathcal{H}_0 t} \mathcal{U} e^{-i\mathcal{H}_0 t} |\phi_{m-1}(t)\rangle,$$

which leads to

$$\begin{aligned} e^{-i\mathcal{H}_0 t} |\phi_1(t)\rangle &= \sum_{n \neq 0} \left(\frac{e^{-iE_n t} - 1}{E_n} - \frac{e^{-iE_n t} - e^{-\epsilon t}}{E_n + i\epsilon} \right) W_{n0} |n\rangle \equiv \sum_{n \neq 0} W_{n0} \Lambda_\epsilon(E_n; t) |n\rangle, \\ e^{-i\mathcal{H}_0 t} |\phi_2(t)\rangle &= \left\{ \sum_{n \neq 0} |W_{n0}|^2 \left[\frac{1}{E_n^2 + \epsilon^2} - \frac{1}{E_n^2} + \frac{e^{-iE_n t}}{E_n} \left(\frac{1}{E_n} - \frac{1}{E_n + i\epsilon} \right) - \frac{e^{-i(E_n - i\epsilon)t}}{E_n - i\epsilon} \left(\frac{1}{E_n} - \frac{1}{E_n + i\epsilon} \right) \right. \right. \\ &\quad \left. \left. + i \frac{1}{E_n} \left(t - \frac{1 - e^{-\epsilon t}}{\epsilon} \right) - i \frac{1}{E_n + i\epsilon} \frac{(1 - e^{-\epsilon t})^2}{2\epsilon} \right] \right\} |0\rangle \\ &\quad + \sum_{n \neq 0} |n\rangle \sum_{m \neq 0, n} W_{nm} W_{m0} \left\{ \frac{1}{E_m} \left[\frac{e^{-iE_n t} - e^{-iE_m t}}{E_n - E_m} - \frac{e^{-iE_n t} - e^{-iE_m t - \epsilon t}}{E_n - E_m + i\epsilon} \right] \right. \\ &\quad \left. - \frac{1}{E_m} \left[\frac{e^{-iE_n t} - 1}{E_n} - \frac{e^{-iE_n t} - e^{-\epsilon t}}{E_n + i\epsilon} \right] \right. \\ &\quad \left. - \frac{1}{E_m + i\epsilon} \left[\frac{e^{-iE_n t} - e^{-iE_m t}}{E_n - E_m} - \frac{e^{-iE_n t} - e^{-iE_m t - \epsilon t}}{E_n - E_m + i\epsilon} \right] + \frac{1}{E_m + i\epsilon} \left[\frac{e^{-iE_n t} - e^{-\epsilon t}}{E_n + i\epsilon} - \frac{e^{-iE_n t} - e^{-2\epsilon t}}{E_n + 2i\epsilon} \right] \right\} \\ &\equiv (A(t) - 1) |0\rangle + \sum_{m \neq 0, n} \sum_{n \neq 0} W_{mn} W_{n0} \Xi_\epsilon(E_m, E_n; t) |m\rangle, \end{aligned} \quad (5)$$

with $W_{nm} = \langle n | \mathcal{U} | m \rangle$.

In the specific case of the Hubbard interaction,

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{U(t)}{V} \sum_{\mathbf{k}\mathbf{p}} \sum_{\mathbf{Q} \neq 0} c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{p}+\mathbf{Q}\downarrow}^\dagger c_{\mathbf{p}\downarrow} c_{\mathbf{k}+\mathbf{Q}\uparrow} = \mathcal{H}_0 + U(t)\mathcal{U}, \quad (7)$$

where the $\mathbf{Q} = \mathbf{0}$ term is not included so to fulfil $W_{nn} = 0, \forall |n\rangle$, the matrix element W_{n0} , where $|0\rangle$ is the unperturbed Fermi sea, is finite only if

$$|n\rangle = c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{p}+\mathbf{Q}\downarrow}^\dagger c_{\mathbf{p}\downarrow} c_{\mathbf{k}+\mathbf{Q}\uparrow} |0\rangle, \quad (8)$$

where $|\mathbf{k}| > k_F$ and $|\mathbf{p} + \mathbf{Q}| > k_F$, hence refer to particles, while $|\mathbf{k} + \mathbf{Q}| \leq k_F$ and $|\mathbf{p}| \leq k_F$, hence refer to holes. The energy of this state is $E_n = \omega_{\mathbf{k}, \mathbf{k}+\mathbf{Q}} + \omega_{\mathbf{p}+\mathbf{Q}, \mathbf{p}} = (\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{Q}}) + (\epsilon_{\mathbf{p}+\mathbf{Q}} - \epsilon_{\mathbf{p}}) > 0$, where energies are measured with respect to the chemical potential. The matrix element is $W_{n0} = 1/V$. At second order we must account also for states with three and four particle-hole pairs besides those with two.

We write the wavefunction up to second order as

$$|\Psi(t)\rangle = A(t) |0\rangle + |\psi_1(t)\rangle + |\psi_2(t)\rangle, \quad (9)$$

where $A(t) = 1 + O(U^2)$, $\langle 0 | \psi_1(t) \rangle = 0 = \langle 0 | \psi_2(t) \rangle$ and

$$|A(t)|^2 + \langle \psi_1(t) | \psi_1(t) \rangle = 1 + O(U^3). \quad (10)$$

We are interested in calculating up to second order the average of the structure-factor operator $\mathcal{S}_{\mathbf{q}} = n_{\mathbf{q}} n_{-\mathbf{q}}/V$ at $\mathbf{q} \neq \mathbf{0}$, which reads, through Eqs. (10), (5) and (6),

$$S(\mathbf{q}, t) \simeq S_0(\mathbf{q}) + \left(\langle \psi_1(t) | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right) + \left(\langle \psi_2(t) | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right) + \langle \psi_1(t) | \mathcal{S}_{\mathbf{q}} - S_0(\mathbf{q}) | \psi_1(t) \rangle$$

$$\begin{aligned}
&= S_0(\mathbf{q}) + U \sum_{n \neq 0} \left[W_{0n} \Lambda_\epsilon(E_n, t)^* \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right] + U^2 \sum_{n \neq 0, m} \sum_{m \neq 0} \left[W_{0m} W_{mn} \Xi_\epsilon(E_n, E_m; t)^* \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right] \\
&\quad + U^2 \sum_{n, m \neq 0} W_{0m} W_{n0} \Lambda_\epsilon(E_m; t)^* \Lambda_\epsilon(E_n; t) \langle m | \mathcal{S}_{\mathbf{q}} - S_0(\mathbf{q}) | n \rangle
\end{aligned} \tag{11}$$

where $S_0(\mathbf{q}) = \langle 0 | \mathcal{S}_{\mathbf{q}} | 0 \rangle$ is the structure factor of the Fermi sea.

The equilibrium perturbation theory can be recovered in the adiabatic limit, which amounts in Eqs. (5) and (6) to set first $\epsilon t \gg 1$ and then send $\epsilon \rightarrow 0$. Indeed, in this limit we obtain

$$\Lambda_{\epsilon \rightarrow 0}(E_n; t) = -\frac{1}{E_n}, \tag{12}$$

$$\Xi_{\epsilon \rightarrow 0}(E_n, E_m; t) = \frac{1}{E_m E_n}, \tag{13}$$

which are the well known expansion coefficients at second order and at equilibrium. Therefore the equilibrium perturbative expansion of the structure factor is

$$\begin{aligned}
S_{\text{eq}}(\mathbf{q}) &= \lim_{\epsilon \rightarrow 0} S(\mathbf{q}, t \gg 1/\epsilon) = S_0(\mathbf{q}) - U \sum_{n \neq 0} \left[\frac{W_{0n}}{E_n} \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right] + U^2 \sum_{n \neq 0, m} \sum_{m \neq 0} \left[\frac{W_{0m} W_{mn}}{E_n E_m} \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right] \\
&\quad + U^2 \sum_{n, m \neq 0} \frac{W_{0m} W_{n0}}{E_n E_m} \langle m | \mathcal{S}_{\mathbf{q}} - S_0(\mathbf{q}) | n \rangle,
\end{aligned} \tag{14}$$

and we know it does not contain any singular term.

However, we are actually interested in the opposite limit of a sudden switch, which amounts to send $\epsilon \rightarrow \infty$ so that

$$\Lambda_{\epsilon \rightarrow \infty}(E_n; t) = -\frac{1 - e^{-iE_n t}}{E_n}, \tag{15}$$

$$\Xi_{\epsilon \rightarrow \infty}(E_n, E_m; t) = \frac{1}{E_m} \frac{e^{-iE_n t} - e^{-iE_m t}}{E_n - E_m} - \frac{1}{E_m} \frac{e^{-iE_n t} - 1}{E_n}. \tag{16}$$

By means of Eqs. (11) and (14), and observing that all matrix elements are real, we can write the structure factor in the sudden limit as

$$\begin{aligned}
S(\mathbf{q}, t) &= S_{\text{eq}}(\mathbf{q}) + U \sum_{n \neq 0} \left[W_{0n} \left(\Lambda_{\epsilon \rightarrow \infty}(E_n, t)^* - \Lambda_0(E_n, t)^* \right) \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right] \\
&\quad + U^2 \sum_{n \neq 0, m} \sum_{m \neq 0} \left[W_{0m} W_{mn} \left(\Xi_{\epsilon \rightarrow \infty}(E_m, E_n; t)^* - \Xi_0(E_m, E_n; t)^* \right) \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle + c.c. \right] \\
&\quad + U^2 \sum_{n, m \neq 0} W_{0m} W_{n0} \left(\Lambda_{\epsilon \rightarrow \infty}(E_m; t)^* \Lambda_{\epsilon \rightarrow \infty}(E_n; t) - \Lambda_0(E_m; t)^* \Lambda_0(E_n; t) \right) \langle m | \mathcal{S}_{\mathbf{q}} - S_0(\mathbf{q}) | n \rangle \\
&= S_{\text{eq}}(\mathbf{q}) + 2U \sum_{n \neq 0} W_{0n} \frac{\cos(E_n t)}{E_n} \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle
\end{aligned} \tag{17}$$

$$+ 2U^2 \sum_{n \neq 0, m} \sum_{m \neq 0} W_{0m} W_{mn} \left(\frac{1}{E_m} \frac{\cos(E_n t) - \cos(E_m t)}{E_n - E_m} - \frac{1}{E_m} \frac{\cos(E_n t)}{E_n} \right) \langle n | \mathcal{S}_{\mathbf{q}} | 0 \rangle \tag{18}$$

$$+ 2U^2 \sum_{n, m \neq 0} W_{0m} W_{n0} \frac{\cos(E_m - E_n)t - \cos(E_m t) - \cos(E_n t)}{E_n E_m} \langle m | \mathcal{S}_{\mathbf{q}} - S_0(\mathbf{q}) | n \rangle. \tag{19}$$

In the thermodynamic limit the second term in Eq. (17) can be written as

$$\delta S^{(1)}(\mathbf{q}, t) = 4U \iint_0^\infty d\omega_1 d\omega_2 \rho_{\mathbf{q}}(\omega_1) \rho_{\mathbf{q}}(\omega_2) \frac{\cos(\omega_1 + \omega_2)t}{\omega_1 + \omega_2}, \tag{20}$$

where $\rho_{\mathbf{q}}(\omega)$ is the density of states of a particle-hole excitation at momentum transferred \mathbf{q} . The denominator vanishes at small frequencies but this singularity is canceled by the numerator since, for $\omega \ll v_F q$, $\rho_{\mathbf{q}}(\omega) \sim \theta(2k_F - q) \omega / v_F q$. As a result, in the long time limit Eq. (20) vanishes with a power law in $1/t$, so that the steady-state $S_*(\mathbf{q})$ after a sudden quench coincides with the equilibrium $S_{\text{eq}}(\mathbf{q})$ up to first order.

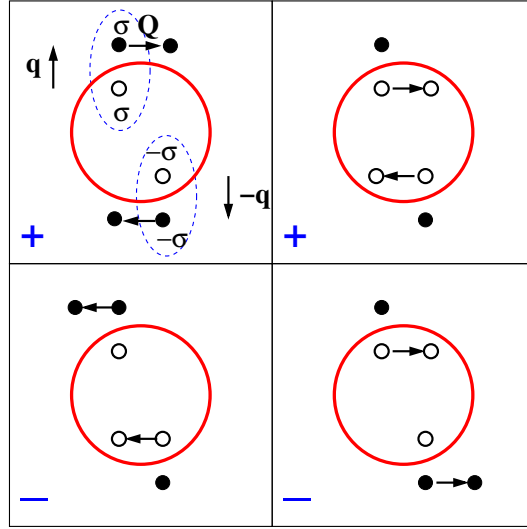


FIG. 1. (Color online) Scattering processes that refer to Eq. (18). The circle represents the Fermi sphere, open dots holes and solid ones particles. The initial particle-hole pairs in $|n\rangle$ are encircled and have opposite spins. The scattering process, i.e. the motion of particles/holes indicated by the arrows, leads to a new state $|m\rangle$.

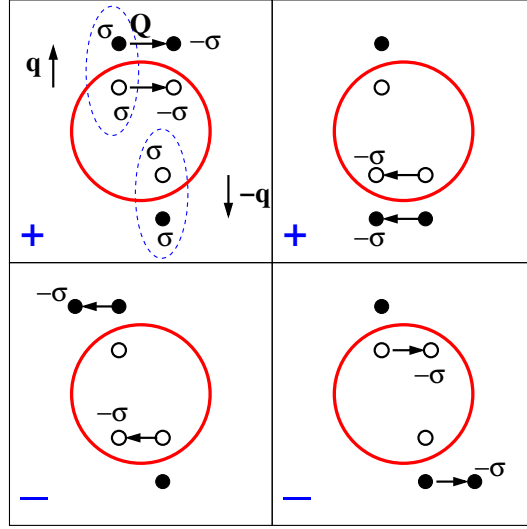


FIG. 2. (Color online) Same as Fig. 1 but for initial pairs that have the same spin.

The explicit evaluation of Eqs. (18) and (19) as well as their final expressions are simple though quite lengthy. Therefore we prefer to show graphically all terms that contribute. Their matrix elements have all the same absolute value, equal to $1/V^3$, apart from a sign that is indicated in the figures.

Let us first consider the term in Eq. (18). The state $|n\rangle$ includes two particle-hole (p-h) pairs at momentum transferred \mathbf{q} and $-\mathbf{q}$ and spin σ and σ' . The intermediate state $|m\rangle$ can be reached by the interaction both from $|n\rangle$ and from $|0\rangle$, hence it contains two p-h pairs with opposite spin and transferred momenta. If $\sigma' = -\sigma$, the processes that bring $|n\rangle$ to $|m\rangle$ are shown in Fig. 1. In the figure $|m\rangle$ contains two opposite-spin p-h pairs at momenta $\mathbf{q} + \mathbf{Q}$ and $-\mathbf{q} - \mathbf{Q}$.

If $\sigma = \sigma'$, the interaction must also flip two spins so to lead to two opposite-spin p-h pairs. The processes are shown in Fig. 2 with their signs.

The term in Eq. (19) has contributions whenever two states $|n\rangle$ and $|m\rangle$, each that contains two p-h pairs with opposite spin and momentum, can be connected by $\mathcal{S}_{\mathbf{q}}$. In particular, the processes generated by $n_{\mathbf{q}\sigma} n_{-\mathbf{q}-\sigma}$ are shown in Fig. 3, where the states $|n\rangle$ and $|m\rangle$ have two p-h pairs at opposite momenta $\pm\mathbf{Q}$ and $\pm(\mathbf{Q} + \mathbf{q})$, respectively.

All processes that are produced by $n_{\mathbf{q}\sigma} n_{-\mathbf{q}\sigma}$ are instead drawn in Figs. 4 and 5. In particular, the lower two panels

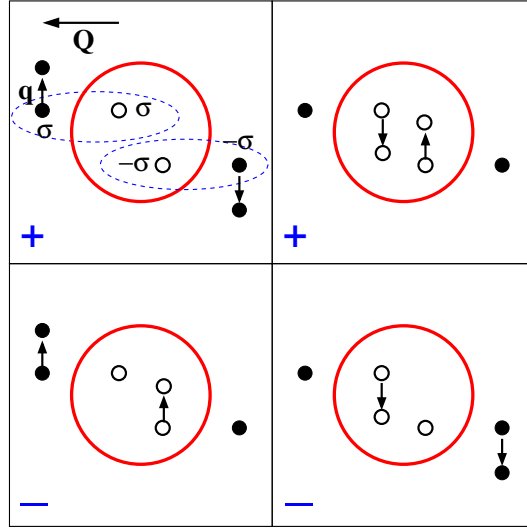


FIG. 3. (Color online) Scattering processes that refer to Eq. (19) and are produced by $n_{\mathbf{q}\sigma} n_{-\mathbf{q}-\sigma}$. The sign of the process is indicated and the initial particle-hole pairs in $|n\rangle$ are encircled.

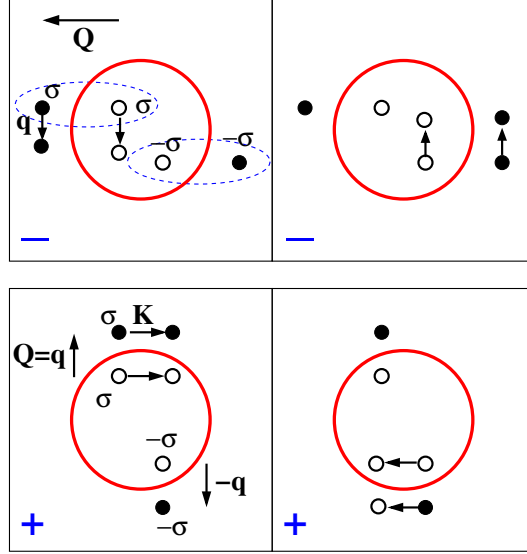


FIG. 4. (Color online) Scattering processes that refer to Eq. (19) and are produced by $n_{\mathbf{q}\sigma} n_{-\mathbf{q}-\sigma}$. The sign of the process is indicated and the initial particle-hole pairs in $|n\rangle$ are encircled.

of Fig. 4 refer to the case in which $|n\rangle$ has two p-h pairs already at momenta $\pm\mathbf{q}$; the interaction simply shifts rigidly one of the pair in momentum space, in the figure the momentum shift is \mathbf{K} .

Finally, Fig. 5 refers to the case in which $|m\rangle = |n\rangle$. Here for instance a particle is shifted by \mathbf{q} and then comes back to the initial position.

Since all states that contribute to Eqs. (18) and (19) have two p-h pairs, their linearly vanishing densities of states at low energy compensate the singularity of the denominators. Therefore the sums, which become integral over a continuous spectrum of degrees of freedom, are all convergent. Even more, all terms but one vanish as a power in $1/t$ for large times. In fact, even though in the sense of distributions,

$$\frac{\cos Et - \cos E't}{E'(E - E')} \xrightarrow{t \rightarrow \infty} -\delta(E) \delta(E' - E) + O\left(\frac{1}{t}\right),$$

$$\frac{\cos(E - E')t}{EE'} \xrightarrow{t \rightarrow \infty} \delta(E) \delta(E') + O\left(\frac{1}{t}\right),$$

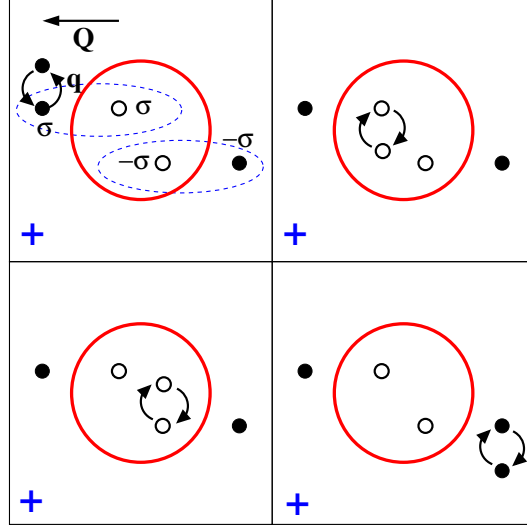


FIG. 5. (Color online) Scattering processes that refer to Eq. (19) and are produced by $n_{\mathbf{q}\sigma} n_{-\mathbf{q}\sigma}$ in the case in which $|n\rangle = |m\rangle$. The sign of the process is indicated and the initial particle-hole pairs in $|n\rangle$ are encircled.

those singularities are killed by the vanishing density of states of p-h excitations, leading to a null result for $t \rightarrow \infty$.

The only term that actually survives is the one shown in Fig. 5, which corresponds to the case $|n\rangle = |m\rangle$ in Eq. (19). We observe that, if we send $|\mathbf{q}| \rightarrow 0$ before taking the limit $t \rightarrow \infty$, this term would be canceled by those shown in the upper panels of Fig. 4. This simply reflects the trivial fact the $S(\mathbf{q} = \mathbf{0}, t) = 0$. The correct procedure is instead to first reach the steady-state $t \rightarrow \infty$, and only after send $|\mathbf{q}| \rightarrow 0$. In this way we do find a finite steady-state contribution from Eqs. (18) and (19), which reads

$$\delta S_*^{(2)}(\mathbf{q}) = \frac{8U^2}{V^3} \sum_{\mathbf{k}\mathbf{p}\mathbf{Q}} n_{\mathbf{k}} \bar{n}_{\mathbf{k}+\mathbf{Q}} n_{\mathbf{p}+\mathbf{Q}} \bar{n}_{\mathbf{p}} \left\{ \frac{\bar{n}_{\mathbf{k}+\mathbf{Q}+\mathbf{q}}}{(\omega_{\mathbf{k}+\mathbf{Q},\mathbf{k}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})(\omega_{\mathbf{k}+\mathbf{Q}+\mathbf{q},\mathbf{k}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})} + \frac{n_{\mathbf{k}-\mathbf{q}}}{(\omega_{\mathbf{k}+\mathbf{Q},\mathbf{k}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})(\omega_{\mathbf{k}+\mathbf{Q},\mathbf{k}-\mathbf{q}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})} \right\}, \quad (21)$$

where $n_{\mathbf{k}} = \theta(k_F - |\mathbf{k}|)$ is the momentum distribution of the Fermi sea and $\bar{n}_{\mathbf{k}} = 1 - n_{\mathbf{k}}$. If we now take the limit $\mathbf{q} \rightarrow \mathbf{0}$, we find a term $\delta S_*^{(2)}(\mathbf{0})$ that is finite plus a correction that starts linear in $|\mathbf{q}|$. Specifically, since $n_{\mathbf{k}}^2 = n_{\mathbf{k}}$,

$$\delta S_*^{(2)}(\mathbf{0}) = \frac{16U^2}{V^3} \sum_{\mathbf{k}\mathbf{p}\mathbf{Q}} \frac{n_{\mathbf{k}} \bar{n}_{\mathbf{k}+\mathbf{Q}} n_{\mathbf{p}+\mathbf{Q}} \bar{n}_{\mathbf{p}}}{(\omega_{\mathbf{k}+\mathbf{Q},\mathbf{k}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})^2} = 16U^2 \int \frac{d\mathbf{Q}}{(2\pi)^3} \iint_0 d\omega_1 d\omega_2 \frac{\rho_{\mathbf{Q}}(\omega_1) \rho_{\mathbf{Q}}(\omega_2)}{(\omega_1 + \omega_2)^2}, \quad (22)$$

and is finite, i.e. not singular. The leading correction to $\delta S_*^{(2)}(\mathbf{0})$ reads

$$\begin{aligned} \delta S_*^{(2)}(\mathbf{q}) - \delta S_*^{(2)}(\mathbf{0}) &= -\frac{16U^2}{V^3} \sum_{\mathbf{k}\mathbf{p}\mathbf{Q}} \frac{\bar{n}_{\mathbf{k}+\mathbf{Q}} n_{\mathbf{p}+\mathbf{Q}} \bar{n}_{\mathbf{p}}}{(\omega_{\mathbf{k}+\mathbf{Q},\mathbf{k}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})(\omega_{\mathbf{k}+\mathbf{Q},\mathbf{k}-\mathbf{q}} + \omega_{\mathbf{p},\mathbf{p}+\mathbf{Q}})} n_{\mathbf{k}} \bar{n}_{\mathbf{k}-\mathbf{q}} \\ &\simeq -|\mathbf{q}| \frac{U^2}{2} \int \frac{d\mathbf{Q}}{(2\pi)^3} \frac{1}{Q} \int_{\text{Max}[0, Q(Q-2)]}^{Q(2+Q)} d\omega_1 \int_0 d\omega_2 \rho_{\mathbf{Q}}(\omega_2) \frac{1}{(\omega_1 + \omega_2)^2} + O(q^2). \end{aligned} \quad (23)$$

Therefore the second order correction to the coefficient of the non-analytic term $\propto |\mathbf{q}|$ in the steady-state differs from that at equilibrium.

DIAGONALIZATION OF THE EFFECTIVE HAMILTONIAN

In this section we explicitly diagonalize the Hamiltonian

$$\mathcal{H}_* = \sum_{\mathbf{Q}} \mathcal{H}_{\mathbf{Q}} = \sum_{\mathbf{Q}} \left[\sum_{\mathbf{k}} \omega_{\mathbf{k},\mathbf{Q}} \left(a_{\mathbf{k},\mathbf{Q}}^\dagger a_{\mathbf{k},\mathbf{Q}} + b_{\mathbf{k},\mathbf{Q}}^\dagger b_{\mathbf{k},\mathbf{Q}} \right) + \frac{U}{V} \sum_{\mathbf{k},\mathbf{p}} \left(b_{\mathbf{k},\mathbf{Q}}^\dagger - a_{\mathbf{k},\mathbf{Q}} \right) \left(b_{\mathbf{p},\mathbf{Q}} - a_{\mathbf{p},\mathbf{Q}}^\dagger \right) \right], \quad (24)$$

relaxing the hard-core constraint. In Eq. (24) $b_{\mathbf{k},\mathbf{Q}}^\dagger$ creates two particles outside the Fermi sea with momenta \mathbf{k} , spin \uparrow and $-\mathbf{k} + \mathbf{Q}$, spin \downarrow , i.e. \mathbf{Q} is the total momentum of the pair. On the contrary, $a_{\mathbf{k},\mathbf{Q}}^\dagger$ creates two holes within the Fermi sea, one at momentum \mathbf{k} , spin \uparrow , and the other at momentum $-\mathbf{k} + \mathbf{Q}$, spin \downarrow . We recall that \mathcal{H}_* describes the excitations of the Fermi sea brought by the interaction in the dilute limit. Hereafter we shall use dimensionless units in which momentum is in units of k_F , energy in units of $\epsilon_F = \hbar^2 k_F^2 / 2m$, and time in units of \hbar / ϵ_F . In addition, the constraint of being outside or inside the Fermi sea will be implicitly hidden in the definition of $b_{\mathbf{k},\mathbf{Q}}$ and $a_{\mathbf{k},\mathbf{Q}}$, respectively.

We start by noting that a pair of holes requires $Q \leq 2$ ($2k_F$ in dimensional units), so that

$$H_* = \sum_{\mathbf{k},\mathbf{Q}:|\mathbf{Q}|\leq 2} \omega_{\mathbf{k},\mathbf{Q}} a_{\mathbf{k},\mathbf{Q}}^\dagger a_{\mathbf{k},\mathbf{Q}} + \omega_{\mathbf{k},\mathbf{Q}} b_{\mathbf{k},\mathbf{Q}}^\dagger b_{\mathbf{k},\mathbf{Q}} + \frac{U}{V} \sum_{\mathbf{k},\mathbf{p},\mathbf{Q}:|\mathbf{Q}|\leq 2} \left(b_{\mathbf{p},\mathbf{Q}}^\dagger - a_{\mathbf{p},\mathbf{Q}} \right) \left(b_{\mathbf{k},\mathbf{Q}} - a_{\mathbf{k},\mathbf{Q}}^\dagger \right) \quad (25)$$

$$+ \sum_{\mathbf{k},\mathbf{Q}:|\mathbf{Q}|>2} \omega_{\mathbf{k},\mathbf{Q}} b_{\mathbf{k},\mathbf{Q}}^\dagger b_{\mathbf{k},\mathbf{Q}} + \frac{U}{V} \sum_{\mathbf{k},\mathbf{p},\mathbf{Q}:|\mathbf{Q}|>2} b_{\mathbf{p},\mathbf{Q}}^\dagger b_{\mathbf{k},\mathbf{Q}}.$$

The last two terms are diagonalized by a simple unitary transformation, while the former two by a generalized canonical transformation, which we shall focus on first.

Let us therefore diagonalize H_* in a subspace at fixed \mathbf{Q} such that $Q \leq 2$. For simplicity we shall drop the label \mathbf{Q} . We define the canonical transformation

$$a_{\mathbf{k}} = \sum_{\epsilon} U_{\mathbf{k}\epsilon} \alpha_{\epsilon} + \sum_{\bar{\epsilon}} V_{\mathbf{k}\bar{\epsilon}} \beta_{\bar{\epsilon}}^\dagger,$$

$$b_{\mathbf{k}}^\dagger = \sum_{\bar{\epsilon}} Z_{\mathbf{k}\bar{\epsilon}} \beta_{\bar{\epsilon}}^\dagger + \sum_{\epsilon} W_{\mathbf{k}\epsilon} \alpha_{\epsilon},$$

where \hat{U} and \hat{Z} are square real matrices, while \hat{V} and \hat{W} are still real but in general rectangular – the number of holes being much smaller than the number of particles in the low-density limit – satisfying

$$U U^T - V V^T = I,$$

$$Z Z^T - W W^T = I,$$

$$U W^T - V Z^T = 0,$$

$$U^T U - W^T W = I,$$

$$Z^T Z - V^T V = I,$$

$$U^T V - W^T Z = 0.$$

The inverse transformation then reads

$$\alpha_{\epsilon} = \sum_{\mathbf{k}} U_{\mathbf{k}\epsilon} a_{\mathbf{k}} - \sum_{\mathbf{k}} W_{\mathbf{k}\epsilon} b_{\mathbf{k}}^\dagger,$$

$$\beta_{\bar{\epsilon}}^\dagger = \sum_{\mathbf{k}} Z_{\mathbf{k}\bar{\epsilon}} b_{\mathbf{k}}^\dagger - \sum_{\mathbf{k}} V_{\mathbf{k}\bar{\epsilon}} a_{\mathbf{k}}.$$

We define the matrix A with elements

$$A_{\mathbf{k}\mathbf{p}} = \omega_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{p}} + \frac{U}{V},$$

for $|\mathbf{k}| \leq 1$ and $|\mathbf{k} - \mathbf{Q}| \leq 1$ ($\leq k_F$ in dimensional units), the matrix B with elements

$$B_{\mathbf{k}\mathbf{p}} = \omega_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{p}} + \frac{U}{V},$$

for $|\mathbf{k}| > 1$ and $|\mathbf{k} - \mathbf{Q}| > 1$, and finally the matrix D with elements

$$D_{\mathbf{k}\mathbf{p}} = -\frac{U}{V},$$

which couples the interior to the exterior of the Fermi sphere. The interaction U is also measured in units of ϵ_F . With the above definitions, the diagonalization of the Hamiltonian corresponds to the eigenvalue equation

$$\begin{pmatrix} A & D \\ -D^T & -B \end{pmatrix} \begin{pmatrix} U & V \\ W & Z \end{pmatrix} = \begin{pmatrix} U & V \\ W & Z \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & -\bar{\epsilon} \end{pmatrix}$$

The solution can be readily found. Specifically, the eigenvalues satisfy

$$1 = \frac{U}{V} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{\epsilon - \omega_{\mathbf{k}}} - \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{\epsilon + \omega_{\mathbf{k}}} \equiv U \chi(\epsilon), \quad (26)$$

$$1 = \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{\bar{\epsilon} - \omega_{\mathbf{k}}} - \frac{U}{V} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{\bar{\epsilon} + \omega_{\mathbf{k}}} = U \chi(-\bar{\epsilon}), \quad (27)$$

where the suffix IN means $|\mathbf{k}| \leq 1$ and $|\mathbf{k} + \mathbf{Q}| \leq 1$, while OUT refers to $|\mathbf{k}| > 1$ and $|\mathbf{k} + \mathbf{Q}| > 1$. In fact, one can solve for any $x \leq 0$,

$$U \chi(x) = \frac{U}{V} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{x - \omega_{\mathbf{k}}} - \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{x + \omega_{\mathbf{k}}} = 1, \quad (28)$$

and set the positive solutions to ϵ and the negative ones to $-\bar{\epsilon}$. The coefficients of the canonical transformation read

$$\begin{aligned} U_{\mathbf{k}\epsilon} &= \sqrt{\frac{U}{V}} N_{\epsilon} \frac{1}{\epsilon - \omega_{\mathbf{k}}}, \\ W_{\mathbf{k}\epsilon} &= \sqrt{\frac{U}{V}} N_{\epsilon} \frac{1}{\epsilon + \omega_{\mathbf{k}}}, \\ Z_{\mathbf{k}\bar{\epsilon}} &= \sqrt{\frac{U}{V}} N_{\bar{\epsilon}} \frac{1}{\bar{\epsilon} - \omega_{\mathbf{k}}}, \\ V_{\mathbf{k}\bar{\epsilon}} &= \sqrt{\frac{U}{V}} N_{\bar{\epsilon}} \frac{1}{\bar{\epsilon} + \omega_{\mathbf{k}}}, \end{aligned}$$

with the parameters N_{ϵ} and $N_{\bar{\epsilon}}$ that should be determined by imposing the transformation to be indeed canonical, i.e.

$$\begin{aligned} 1 &= N_{\epsilon}^2 \frac{U}{V} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{(\epsilon - \omega_{\mathbf{k}})^2} - N_{\epsilon}^2 \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{k}})^2}, \\ 1 &= N_{\bar{\epsilon}}^2 \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{(\bar{\epsilon} - \omega_{\mathbf{k}})^2} - N_{\bar{\epsilon}}^2 \frac{U}{V} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{k}})^2}, \\ 0 &= N_{\epsilon} N_{\bar{\epsilon}} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{(\epsilon - \omega_{\mathbf{k}})(\bar{\epsilon} + \omega_{\mathbf{k}})} - N_{\epsilon} N_{\bar{\epsilon}} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{(\bar{\epsilon} - \omega_{\mathbf{k}})(\epsilon + \omega_{\mathbf{k}})}. \end{aligned}$$

We observe that the last condition is also equivalent to

$$0 = N_{\epsilon} N_{\bar{\epsilon}} \sum_{\mathbf{k}}^{\text{IN}} \frac{1}{(\epsilon - \omega_{\mathbf{k}})(\bar{\epsilon} + \omega_{\mathbf{k}})} - N_{\epsilon} N_{\bar{\epsilon}} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{(\bar{\epsilon} - \omega_{\mathbf{k}})(\epsilon + \omega_{\mathbf{k}})} = V \frac{N_{\epsilon} N_{\bar{\epsilon}}}{\epsilon + \bar{\epsilon}} [\chi(\epsilon) - \chi(-\bar{\epsilon})] = 0,$$

hence is automatically satisfied because of the eigenvalue equations (26) and (27).

We conclude by noting that the same calculation can be carried out also for $Q > 2$. In this case there are no a -bosons, and one only needs to find the unitary transformation $Z_{\mathbf{k}\bar{\epsilon}}$, i.e. one has to solve

$$1 = \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{\bar{\epsilon} - \omega_{\mathbf{k}}}, \quad (29)$$

and define

$$Z_{\mathbf{k}\bar{\epsilon}} = \sqrt{\frac{U}{V}} N_{\bar{\epsilon}} \frac{1}{\bar{\epsilon} - \omega_{\mathbf{k}}},$$

where

$$N_{\bar{\epsilon}}^{-2} = \frac{U}{V} \sum_{\mathbf{k}}^{\text{OUT}} \frac{1}{(\bar{\epsilon} - \omega_{\mathbf{k}})^2}.$$

Continuum limit

A proper definition of a steady-state requires to take first the thermodynamic limit, hence to turn all finite sums into integrals over a continuum of degrees of freedom, and only afterwards send the time $t \rightarrow \infty$.

In order to understand what the above formulas mean in the continuum limit, we may follow the different route to solve the problem at equilibrium within the Matsubara technique. We would find for instance that the imaginary-time Fourier transforms of $G_b(\tau, \mathbf{k}) = -\langle T_\tau (b_{\mathbf{k}}(\tau) b_{\mathbf{k}}^\dagger) \rangle$ and $G_a(\tau, \mathbf{k}) = -\langle T_\tau (a_{\mathbf{k}}(\tau) a_{\mathbf{k}}^\dagger) \rangle$ read

$$G_b(i\Omega, \mathbf{k}) = \frac{1}{i\Omega - \omega_{\mathbf{k}}} + \frac{1}{V} \frac{T(-i\Omega)}{(i\Omega - \omega_{\mathbf{k}})^2},$$

$$G_a(i\Omega, \mathbf{k}) = \frac{1}{i\Omega - \omega_{\mathbf{k}}} + \frac{1}{V} \frac{T(i\Omega)}{(i\Omega - \omega_{\mathbf{k}})^2},$$

where

$$T(i\Omega) = \frac{U}{1 - U \chi(i\Omega)}, \quad (30)$$

is the usual definition of the T -matrix, with $\chi(z)$ the Cooper bubble that actually corresponds to the same function defined above continued in the complex frequency plane. In the continuum limit $\chi(z)$ has a branch cut on the real axis, specifically

$$\chi(\epsilon + i0^+) - \chi(\epsilon - i0^+) = -2\pi i \frac{1}{V} \sum_{\mathbf{k}}^{\text{IN}} \delta(\epsilon - \omega_{\mathbf{k}}) + 2\pi i \frac{1}{V} \sum_{\mathbf{k}}^{\text{OUT}} \delta(\epsilon + \omega_{\mathbf{k}}) \equiv -2\pi i \mathcal{N}_{\text{IN}}(\epsilon) + 2\pi i \mathcal{N}_{\text{OUT}}(-\epsilon),$$

where $\mathcal{N}_{\text{IN}}(x)$ and $\mathcal{N}_{\text{OUT}}(x)$ are defined only for $x > 0$ and correspond to the density of states of pairs of holes and particles, respectively, at total momentum \mathbf{Q} . Introducing back the dependence on \mathbf{Q} and in dimensionless units,

$$\begin{aligned} \frac{1}{(k_F a)^3} \mathcal{N}_{\text{IN}}(\epsilon, \mathbf{Q}) &= \theta(Q(2 - Q) - \epsilon) \frac{\epsilon}{16\pi^2 Q} \\ &\quad + \theta(\epsilon - Q(2 - Q)) \theta((2 + Q)(2 - Q) - 2\epsilon) \frac{\sqrt{4 - 2\epsilon - Q^2}}{16\pi^2}, \end{aligned} \quad (31)$$

$$\frac{1}{(k_F a)^3} \mathcal{N}_{\text{OUT}}(\epsilon, \mathbf{Q} \leq 2) = \theta(Q(2 + Q) - \epsilon) \frac{\epsilon}{16\pi^2 Q} + \theta(\epsilon - Q(2 + Q)) \frac{\sqrt{4 + 2\epsilon - Q^2}}{16\pi^2}, \quad (32)$$

$$\begin{aligned} \frac{1}{(k_F a)^3} \mathcal{N}_{\text{OUT}}(\epsilon, \mathbf{Q} > 2) &= \theta(2\epsilon - (Q + 2)(Q - 2)) \theta(Q(Q - 2) - \epsilon) \frac{\sqrt{2\epsilon + 4 - Q^2}}{16\pi^2} \\ &\quad + \theta(\epsilon - Q(2 - Q)) \theta(Q(2 + Q) - \epsilon) \frac{\epsilon}{16\pi^2 Q} \\ &\quad + \theta(\epsilon - Q(2 + Q)) \frac{\sqrt{2\epsilon + 4 - Q^2}}{16\pi^2}, \end{aligned} \quad (33)$$

where a is the lattice spacing.

We then note that, in the limit of zero temperature,

$$\begin{aligned} \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rangle &= -T \sum_{\Omega} e^{i\Omega 0^+} G_b(i\Omega, \mathbf{k}) = \frac{U}{V} \int_0^\infty d\epsilon \frac{1}{(\epsilon + \omega_{\mathbf{k}})^2} \frac{U \mathcal{N}_{\text{IN}}(\epsilon)}{(1 - U \chi'(\epsilon))^2 + \pi^2 U^2 \mathcal{N}_{\text{IN}}(\epsilon)^2}, \\ \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle &= -T \sum_{\Omega} e^{i\Omega 0^+} G_a(i\Omega, \mathbf{k}) = \frac{U}{V} \int_0^\infty d\epsilon \frac{1}{(\epsilon + \omega_{\mathbf{k}})^2} \frac{U \mathcal{N}_{\text{OUT}}(\epsilon)}{(1 - U \chi'(-\epsilon))^2 + \pi^2 U^2 \mathcal{N}_{\text{OUT}}(\epsilon)^2}, \end{aligned}$$

where $\chi'(\epsilon) = \Re \chi(\epsilon - i0^+)$. On the other hand, if we calculate the above average values directly via exact diagonalization, we find

$$\begin{aligned} \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \rangle &= \frac{U}{V} \sum_{\epsilon} N_{\epsilon}^2 \frac{1}{(\epsilon + \omega_{\mathbf{k}})^2}, \\ \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle &= \frac{U}{V} \sum_{\bar{\epsilon}} N_{\bar{\epsilon}}^2 \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{k}})^2}, \end{aligned}$$

showing that, in the continuum limit $\sum_\epsilon \rightarrow \int d\epsilon$,

$$\begin{aligned} U N_\epsilon^2 &\rightarrow \frac{1}{\pi} \Im T(\epsilon - i0^+) = N_{\text{IN}}(\epsilon), \\ U N_{\bar{\epsilon}}^2 &\rightarrow \frac{1}{\pi} \Im T(-\bar{\epsilon} + i0^+) = N_{\text{OUT}}(\bar{\epsilon}). \end{aligned}$$

Time dependent averages

The advantage of the exact diagonalization is to allow calculating the out-of-equilibrium evolution after suddenly switching on U at time $t = 0$, without solving any integral equation. The initial state is thence the vacuum of the original bosons, but the operators are time-evolved with the $U \neq 0$ Hamiltonian. By means of the exact-diagonalization, we find that, for $Q \leq 2$,

$$\begin{aligned} \langle \alpha_\epsilon^\dagger(t) \alpha_{\epsilon'}(t) \rangle &= N_\epsilon N_{\epsilon'} e^{i(\epsilon - \epsilon')t} \frac{U}{V} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{p}})(\epsilon' + \omega_{\mathbf{p}})}, \\ \langle \beta_{\bar{\epsilon}}^\dagger(t) \beta_{\bar{\epsilon}'}(t) \rangle &= N_{\bar{\epsilon}} N_{\bar{\epsilon}'} e^{i(\bar{\epsilon} - \bar{\epsilon}')t} \frac{U}{V} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{p}})(\bar{\epsilon}' + \omega_{\mathbf{p}})}, \\ \langle \alpha_\epsilon^\dagger(t) \beta_{\bar{\epsilon}}^\dagger(t) \rangle &= -N_\epsilon N_{\bar{\epsilon}} e^{i(\epsilon + \bar{\epsilon})t} \frac{U}{V} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{p}})(\bar{\epsilon} - \omega_{\mathbf{p}})}, \\ \langle \beta_{\bar{\epsilon}}^\dagger(t) \alpha_\epsilon^\dagger(t) \rangle &= -N_\epsilon N_{\bar{\epsilon}} e^{i(\epsilon + \bar{\epsilon})t} \frac{U}{V} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{(\epsilon - \omega_{\mathbf{p}})(\bar{\epsilon} + \omega_{\mathbf{p}})}. \end{aligned}$$

All other averages can be obtained by the above ones, as for instance

$$\langle \alpha_{\epsilon'}(t) \alpha_\epsilon^\dagger(t) \rangle = \delta_{\epsilon\epsilon'} + \langle \alpha_\epsilon^\dagger(t) \alpha_{\epsilon'}(t) \rangle,$$

or

$$\langle \alpha_\epsilon(t) \beta_{\bar{\epsilon}}(t) \rangle = \left(\langle \beta_{\bar{\epsilon}}^\dagger(t) \alpha_\epsilon^\dagger(t) \rangle \right)^*.$$

For $Q > 2$ the boson vacuum remains instead unaffected, only the excitation energies are modified by interaction.

It follows therefore that, for $Q \leq 2$,

$$\begin{aligned} \langle a_{\mathbf{k}}^\dagger(t) a_{\mathbf{k}}(t) \rangle &= \frac{U}{V} \sum_{\bar{\epsilon}} N_{\bar{\epsilon}}^2 \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{k}})^2} \\ &+ \frac{U^2}{V^2} \sum_{\epsilon\epsilon'} \cos(\epsilon - \epsilon')t N_\epsilon^2 N_{\epsilon'}^2 \frac{1}{(\epsilon - \omega_{\mathbf{k}})(\epsilon' - \omega_{\mathbf{k}})} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{p}})(\epsilon' + \omega_{\mathbf{p}})} \\ &+ \frac{U^2}{V^2} \sum_{\bar{\epsilon}\bar{\epsilon}'} \cos(\bar{\epsilon} - \bar{\epsilon}')t N_{\bar{\epsilon}}^2 N_{\bar{\epsilon}'}^2 \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{k}})(\bar{\epsilon}' + \omega_{\mathbf{k}})} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{p}})(\bar{\epsilon}' + \omega_{\mathbf{p}})} \\ &- \frac{U^2}{V^2} \sum_{\epsilon\bar{\epsilon}} 2 \cos(\epsilon t + \bar{\epsilon} t) N_\epsilon^2 N_{\bar{\epsilon}}^2 \frac{1}{(\epsilon - \omega_{\mathbf{k}})(\bar{\epsilon} + \omega_{\mathbf{k}})} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{p}})(\bar{\epsilon} - \omega_{\mathbf{p}})}, \\ \langle b_{\mathbf{k}}^\dagger(t) b_{\mathbf{k}}(t) \rangle &= \frac{U}{V} \sum_{\epsilon} N_\epsilon^2 \frac{1}{(\epsilon + \omega_{\mathbf{k}})^2} \\ &+ \frac{U^2}{V^2} \sum_{\bar{\epsilon}\bar{\epsilon}'} \cos(\bar{\epsilon} - \bar{\epsilon}')t N_{\bar{\epsilon}}^2 N_{\bar{\epsilon}'}^2 \frac{1}{(\bar{\epsilon} - \omega_{\mathbf{k}})(\bar{\epsilon}' - \omega_{\mathbf{k}})} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{p}})(\bar{\epsilon}' + \omega_{\mathbf{p}})} \\ &+ \frac{U^2}{V^2} \sum_{\epsilon\epsilon'} \cos(\epsilon - \epsilon')t N_\epsilon^2 N_{\epsilon'}^2 \frac{1}{(\epsilon + \omega_{\mathbf{k}})(\epsilon' + \omega_{\mathbf{k}})} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{p}})(\epsilon' + \omega_{\mathbf{p}})} \end{aligned} \tag{34}$$

$$-\frac{U^2}{V^2} \sum_{\epsilon \bar{\epsilon}} 2 \cos(\epsilon t + \bar{\epsilon} t) N_{\epsilon}^2 N_{\bar{\epsilon}}^2 \frac{1}{(\bar{\epsilon} - \omega_{\mathbf{k}})(\epsilon + \omega_{\mathbf{k}})} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{p}})(\epsilon - \omega_{\mathbf{p}})}, \quad (35)$$

while, for $Q > 2$, $\langle b_{\mathbf{k}}^{\dagger}(t) b_{\mathbf{k}}(t) \rangle = 0$.

The first terms on the right hand sides of Eqs. (34) and (35) are the equilibrium values, hence all the rest is due to the sudden quench. We observe that, because of the eigenvalue equations,

$$\frac{U}{V} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{(\bar{\epsilon} + \omega_{\mathbf{p}})(\epsilon - \omega_{\mathbf{p}})} = \frac{U}{V} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{(\epsilon + \omega_{\mathbf{p}})(\bar{\epsilon} - \omega_{\mathbf{p}})} = \frac{1}{\epsilon + \bar{\epsilon}} \left[1 + \frac{U}{V} \sum_{\mathbf{p}}^{\text{IN}} \frac{1}{\bar{\epsilon} + \omega_{\mathbf{p}}} + \frac{U}{V} \sum_{\mathbf{p}}^{\text{OUT}} \frac{1}{\epsilon + \omega_{\mathbf{p}}} \right], \quad (36)$$

which therefore brings no singularity when $\bar{\epsilon} = \omega_{\mathbf{p}}$.

Since before the continuum limit is taken $T(\omega_{\mathbf{k}}) = 0$, if we consider a contour that run anti-clockwise closely around the positive real axis, then

$$I = U^2 \sum_{\epsilon \epsilon'} N_{\epsilon}^2 N_{\epsilon'}^2 \frac{\cos(\epsilon t - \epsilon' t)}{(\epsilon - \omega_{\mathbf{k}})(\epsilon' - \omega_{\mathbf{k}})} F(\epsilon, \epsilon') = \oint \frac{dz dz'}{(2\pi i)^2} T(z) T(z') \frac{\cos(zt - z't)}{(z - \omega_{\mathbf{k}})(z' - \omega_{\mathbf{k}})} F(z, z'),$$

where $F(z, z') = F(z', z)$ is assumed analytic. We can now take the continuum limit and find that

$$\begin{aligned} I \rightarrow \oint_0 d\epsilon d\epsilon' N_{\text{IN}}(\epsilon) N_{\text{IN}}(\epsilon') \frac{\cos(\epsilon t - \epsilon' t)}{(\epsilon - \omega_{\mathbf{k}})(\epsilon' - \omega_{\mathbf{k}})} F(\epsilon, \epsilon') \\ - 2T'(\omega_{\mathbf{k}}) \oint d\epsilon N_{\text{IN}}(\epsilon) \frac{\cos(\epsilon t - \omega_{\mathbf{k}} t)}{(\epsilon - \omega_{\mathbf{k}})} F(\epsilon, \omega_{\mathbf{k}}) + T'(\omega_{\mathbf{k}})^2 F(\omega_{\mathbf{k}}, \omega_{\mathbf{k}}), \end{aligned}$$

where $\oint d\epsilon (\dots)$ means the Cauchy principal value integration and $T'(\epsilon) = \Re T(\epsilon - i0^+)$. Seemingly,

$$\begin{aligned} \bar{I} = U^2 \sum_{\bar{\epsilon} \bar{\epsilon}'} N_{\bar{\epsilon}}^2 N_{\bar{\epsilon}'}^2 \frac{\cos(\bar{\epsilon} t - \bar{\epsilon}' t)}{(\bar{\epsilon} - \omega_{\mathbf{k}})(\bar{\epsilon}' - \omega_{\mathbf{k}})} F(\bar{\epsilon}, \bar{\epsilon}') = \oint \frac{dz dz'}{(2\pi i)^2} T(-z) T(-z') \frac{\cos(zt - z't)}{(z - \omega_{\mathbf{k}})(z' - \omega_{\mathbf{k}})} F(z, z') \\ \rightarrow \oint_0 d\epsilon d\epsilon' N_{\text{OUT}}(\epsilon) N_{\text{OUT}}(\epsilon') \frac{\cos(\epsilon t - \epsilon' t)}{(\epsilon - \omega_{\mathbf{k}})(\epsilon' - \omega_{\mathbf{k}})} F(\epsilon, \epsilon') - 2T'(-\omega_{\mathbf{k}}) \oint d\epsilon N_{\text{OUT}}(\epsilon) \frac{\cos(\epsilon t - \omega_{\mathbf{k}} t)}{(\epsilon - \omega_{\mathbf{k}})} F(\epsilon, \omega_{\mathbf{k}}) \\ + T'(-\omega_{\mathbf{k}})^2 F(\omega_{\mathbf{k}}, \omega_{\mathbf{k}}). \end{aligned}$$

We are actually interested in the large- t limit. We observe that

$$\lim_{t \rightarrow \infty} \frac{\sin(\epsilon t - \omega_{\mathbf{k}} t)}{\epsilon - \omega_{\mathbf{k}}} = \pi \delta(\epsilon - \omega_{\mathbf{k}}), \quad (37)$$

so that

$$\lim_{t \rightarrow \infty} I = \left[T'(\omega_{\mathbf{k}})^2 + \pi^2 N_{\text{IN}}(\omega_{\mathbf{k}})^2 \right] F(\omega_{\mathbf{k}}, \omega_{\mathbf{k}}) = \left| T(\omega_{\mathbf{k}} - i0^+) \right|^2 F(\omega_{\mathbf{k}}, \omega_{\mathbf{k}}), \quad (38)$$

and

$$\lim_{t \rightarrow \infty} \bar{I} = \left[T'(-\omega_{\mathbf{k}})^2 + \pi^2 N_{\text{OUT}}(\omega_{\mathbf{k}})^2 \right] F(\omega_{\mathbf{k}}, \omega_{\mathbf{k}}) = \left| T(-\omega_{\mathbf{k}} + i0^+) \right|^2 F(\omega_{\mathbf{k}}, \omega_{\mathbf{k}}). \quad (39)$$

Steady state values

We are now in the position to evaluate the steady state value of the Fermi distribution jump $Z(t)$ defined by

$$Z(t) \simeq 1 - \sum_{\mathbf{Q}} \langle \psi(t) | a_{\mathbf{k}, \mathbf{Q}}^{\dagger} a_{\mathbf{k}, \mathbf{Q}} + b_{\mathbf{k}, \mathbf{Q}}^{\dagger} b_{\mathbf{k}, \mathbf{Q}} | \psi(t) \rangle, \quad (40)$$

where $|\mathbf{k}| = k_F$ and $|\psi(t)\rangle$ is the boson-vacuum evolved with the Hamiltonian (24). Once we set $|\mathbf{k}| = k_F$ and integrate over \mathbf{Q} , the only terms in Eqs. (34) and (35) than could survive in the $t \rightarrow \infty$ limit are, apart from the

equilibrium values, the second ones. By the formulas above, we thence find, inserting back the total momentum label \mathbf{Q} , that

$$\langle a_{\mathbf{k},\mathbf{Q}}^\dagger(t) a_{\mathbf{k},\mathbf{Q}}(t) \rangle \rightarrow \frac{1}{V} \int_0^\infty \frac{d\epsilon}{\pi} \frac{\Im T(-\epsilon + i0^+, \mathbf{Q})}{(\epsilon + \omega_{\mathbf{k},\mathbf{Q}})^2} + \frac{1}{V} \left| T(\omega_{\mathbf{k},\mathbf{Q}} - i0^+, \mathbf{Q}) \right|^2 \int d\omega \frac{\mathcal{N}_{\text{OUT}}(\omega, \mathbf{Q})}{(\omega_{\mathbf{k},\mathbf{Q}} + \omega)^2}, \quad (41)$$

$$\langle b_{\mathbf{k},\mathbf{Q}}^\dagger(t) b_{\mathbf{k},\mathbf{Q}}(t) \rangle \rightarrow \frac{1}{V} \int_0^\infty d\epsilon \frac{\Im T(\epsilon - i0^+, \mathbf{Q})}{(\epsilon + \omega_{\mathbf{k},\mathbf{Q}})^2} + \frac{1}{V} \left| T(-\omega_{\mathbf{k},\mathbf{Q}} + i0^+, \mathbf{Q}) \right|^2 \int d\omega \frac{\mathcal{N}_{\text{IN}}(\omega, \mathbf{Q})}{(\omega_{\mathbf{k},\mathbf{Q}} + \omega)^2}, \quad (42)$$

As anticipated, the boson occupation numbers are $\sim 1/V$, thus justifying our discarding the hard-core constraint.

Through Eqs. (40) – (42) above, and by means of Eqs. (46) and (48) below, we find that the steady state value of $Z_* = Z(t \rightarrow \infty)$ reads

$$\begin{aligned} Z_* = Z_{\text{eq.}} - \frac{(k_F a)^3}{8\pi^2} \int_0^2 Q dQ \left\{ \int_0^{Q(2-Q)} d\omega \left| T(\omega - i0^+, \mathbf{Q}) \right|^2 \int_0^\infty d\epsilon \frac{\mathcal{N}_{\text{OUT}}(\epsilon, \mathbf{Q})}{(\omega + \epsilon)^2} \right. \\ \left. + \int_0^{Q(2+Q)} d\omega \left| T(-\omega + i0^+, \mathbf{Q}) \right|^2 \int_0^\infty d\epsilon \frac{\mathcal{N}_{\text{IN}}(\epsilon, \mathbf{Q})}{(\omega + \epsilon)^2} \right\}, \end{aligned} \quad (43)$$

where

$$\begin{aligned} Z_{\text{eq.}} = 1 - \frac{(k_F a)^3}{8\pi^2} \int_0^2 Q dQ \left\{ \int_0^{Q(2-Q)} d\omega \int_0^\infty \frac{d\epsilon}{\pi} \frac{\Im T(-\epsilon + i0^+, \mathbf{Q})}{(\epsilon + \omega)^2} \right. \\ \left. + \int_0^{Q(2+Q)} d\omega \int_0^\infty d\epsilon \frac{\Im T(\epsilon - i0^+, \mathbf{Q})}{(\epsilon + \omega)^2} \right\}, \end{aligned} \quad (44)$$

is the equilibrium value at zero temperature[1] and a the lattice spacing.

Useful formulas

Let us consider a function $F(Q, \omega_{\mathbf{k},\mathbf{Q}})$ and define as $n_{\mathbf{k}}$ the non-interacting momentum distribution. Because of momentum isotropy

$$f_{\text{IN}}(\mathbf{k}) = \frac{1}{V} \sum_{\mathbf{Q}} F(Q, \omega_{\mathbf{k},\mathbf{Q}}) n_{\mathbf{k}} n_{-\mathbf{k}+\mathbf{Q}},$$

only depends on $|\mathbf{k}| = k$. Therefore

$$\begin{aligned} f_{\text{IN}}(k_F) &= \rho_0^{-1} \frac{1}{V} \sum_{\mathbf{k}} f(\mathbf{k}) \delta(\epsilon_{\mathbf{k}}) = \rho_0^{-1} \frac{1}{V} \sum_{\mathbf{Q}} \frac{1}{V} \sum_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) F(Q, \omega_{\mathbf{k},\mathbf{Q}}) n_{\mathbf{k}} n_{-\mathbf{k}+\mathbf{Q}} \\ &= \rho_0^{-1} \frac{1}{V} \sum_{\mathbf{Q}} \int_0^\infty d\omega \rho_{\text{IN}}(\omega, \mathbf{Q}) F(Q, \omega), \end{aligned} \quad (45)$$

where ρ_0 is the non-interacting density of states at the Fermi energy and

$$\rho_{\text{IN}}(\omega) = \frac{1}{V} \sum_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) \delta\left(\omega - \frac{\omega_{\mathbf{k},\mathbf{Q}}}{\epsilon_F}\right) n_{\mathbf{k}} n_{-\mathbf{k}+\mathbf{Q}} = \frac{\rho_0}{4Q} \theta(Q(2-Q) - \omega) \theta(2-Q). \quad (46)$$

Seemingly, if

$$f_{\text{OUT}}(\mathbf{k}) = \frac{1}{V} \sum_{\mathbf{Q}} F(Q, \omega_{\mathbf{k},\mathbf{Q}}) (1 - n_{\mathbf{k}}) (1 - n_{-\mathbf{k}+\mathbf{Q}}),$$

then

$$f_{\text{OUT}}(k_F) = \rho_0^{-1} \frac{1}{V} \sum_{\mathbf{Q}} \int_0^\infty d\omega \rho_{\text{OUT}}(\omega, \mathbf{Q}) F(Q, \omega), \quad (47)$$

where

$$\rho_{\text{OUT}}(\omega) = \frac{1}{V} \sum_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) \delta\left(\omega - \frac{\omega_{\mathbf{k}, \mathbf{Q}}}{\epsilon_F}\right) n_{\mathbf{k}} n_{-\mathbf{k}+\mathbf{Q}} = \frac{\rho_0}{4Q} \theta(Q(2+Q) - \omega). \quad (48)$$

The above expressions are useful to evaluate the momentum distribution jump at k_F .

[1] V.M. Galitskii, Zh. Eksp. Teor. Fiz. **34**, 104 (1958), [Sov. Phys. JETP **7**, 104 (1958)].
